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J80-064

On the Fundamental Solution for a Cascade

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IN the theories of compressor blade flutter, of noise generation by or transmission through fans, and of lift fluctuations on a cascade due to inlet disturbances, one is interested in a certain fundamental solution, say p, to the wave equation. This fundamental solution satisfies

$$\left(\frac{\partial}{\partial t}+U_\infty\frac{\partial}{\partial x}\right)^2 p-c_\infty^2\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)=\sum_{n=-\infty}^{\infty}e^{i\omega t}e^{in\sigma}\delta(x-sn\cos\lambda)\,\delta(y-sn\sin\lambda) \tag{1}$$

where $U_\infty=\text{const}$, $c_\infty=\text{const}$ are the undisturbed flow velocity and speed of sound in the medium, and t and (x,y) denote time and Cartesian space coordinates, respectively (see Fig. 1). The right-hand side of Eq. (1) represents an infinite array of pressure sources placed along the line $y-x\tan\lambda=0$; these sources are separated by a constant distance s and are harmonically oscillating at a circular frequency ω. The phase difference between adjacent sources is σ. [Note: $i=(-1)^{1/2}$ and $\delta=\delta(\xi)$ denotes the delta function.]

The long time solution ($t\to\infty$) to Eq. (1) is expressible as $p=P\exp(i\omega t)$, where P consists of a poorly converging infinite series of Bessel functions. Kaji and Okazaki[1] have shown how to render this infinite series rapidly convergent in the subsonic case (i.e., $U_\infty/c_\infty=M<1$) by a clever and complex application of the Poisson summation formula.

The purpose of this Note is twofold: first to show that the essential results of Kaji and Okazaki may be obtained by a simpler alternate route, and, second, to point out that these results are valid not only in the strictly subsonic case ($M<1$) but also in the subsonic leading edge case, $M\sin\lambda<1$ but $M>1$.

To do this, Eq. (1) is solved with *initial* conditions $p=\partial p/\partial t=0$ at $t=0$. While this procedure apparently implies added complexity, this is not the case because the required sum is evaluated readily. First consider the solution, p_n, for a single source, say the nth one. It is known[2] that

$$p_n=\frac{e^{i\omega t}e^{in\sigma}}{2\pi c_\infty}\int_0^t e^{-i\omega T}\frac{H(\zeta)}{\zeta^{1/2}}\,\mathrm{d}T \tag{2a}$$

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where $\zeta=c_\infty^2T^2-(x-sn\cos\lambda-U_\infty T)^2-(y-sn\sin\lambda)^2$ and by superposition

$$p=\sum_{n=-\infty}^{\infty}p_n \tag{2b}$$

Here $H(\zeta)$ denotes the Heaviside function such that $H=1$ for $\zeta>0$ and $H=0$ for $\zeta<0$.

If now the order of the integration and summation is interchanged in Eq. (2b) after Eq. (2a) is substituted for p_n, then a *finite* sum is first needed evaluation. It is known[3] that

$$\sum_{n=-\infty}^{\infty}\frac{e^{in\sigma}H(\zeta)}{\zeta^{1/2}}=\frac{\pi}{s}e^{-i\sigma\xi/s}\sum_{n=-\infty}^{\infty}e^{2\pi in\xi/s}J_0\{\eta^{1/2}(2\pi n-\sigma)/s\} \tag{3}$$

where

$$\xi=-(x-U_\infty T)\cos\lambda-y\sin\lambda$$

$$\eta=c_\infty^2T^2-[(x-U_\infty T)\sin\lambda-y\cos\lambda]^2$$

and J_0 is a Bessel function. It is emphasized that the sum on the left-hand side of Eq. (3) is finite because of the H function. Finally, as $t\to\infty$, an integral (actually a Fourier transform) with respect to T is left. This is again standard,[4] so the final result is

$$p=\frac{\exp[i\omega(t-\tilde{y}M\sin\lambda/c_\infty\beta)]}{2c_\infty^2}\Bigg[\sum_n\exp[i(2\pi n-\sigma)(\tilde{y}M^2\sin\lambda\cos\lambda/\beta-\tilde{x})/s]\times\frac{\exp[-|\tilde{y}/s\beta|\Gamma^{1/2}]}{\Gamma^{1/2}}$$
$$+\sum_n\exp[i(2\pi n-\sigma)(\tilde{y}M^2\sin\lambda\cos\lambda/\beta-\tilde{x})/s]\times i\{\mathrm{sgn}[(2\pi n-\sigma)M\cos\lambda-\omega s/c_\infty]\cos[|\tilde{y}|(-\Gamma)^{1/2}/s\beta]+i\sin[|\tilde{y}|(-\Gamma)^{1/2}/s\beta]\}/(-\Gamma)^{1/2}\Bigg] \tag{4a}$$

where $\tilde{x}$ and $\tilde{y}$ are the rotated coordinates $\tilde{x}=x\cos\lambda+y\sin\lambda$, $\tilde{y}=-x\sin\lambda+y\cos\lambda$, and $\beta=1-M^2\sin^2\lambda$. Here Γ is the *propagation parameter*

$$\Gamma=(2\pi n-\sigma)^2(1-M^2\sin^2\lambda)-[(2\pi n-\sigma)M\cos\lambda-\omega s/c_\infty]^2 \tag{4b}$$

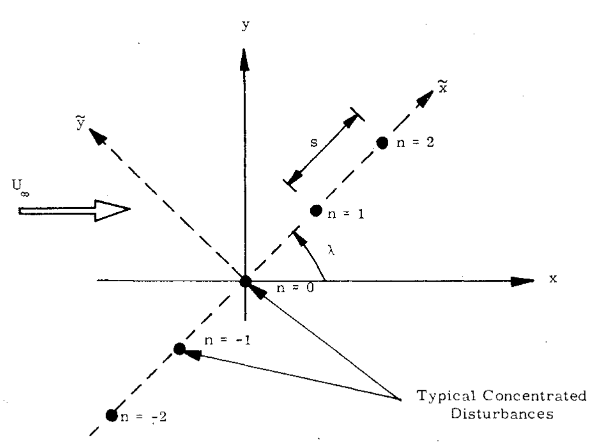

Fig. 1 Geometry of the problem.

and sgn denotes the signum function.† The first sum in Eq. (4a) holds for those integral values of n for which $\Gamma>0$, and the second sum for those n for which $\Gamma<0$. n varies from $(-\infty)$ to $(+\infty)$. As $|n|\to\infty$, $\Gamma\to(2\pi n)^2(1-M^2)$, and in the subsonic case Γ is positive. Thus, when $M<1$, the contributions to the infinite sum from terms for which $|n|$ is large are exponentially small. Thus Eq. (4a) is a rapidly (i.e., exponentially fast) converging series; this series is quite suitable for numerical computation. Second, Eq. (4a) is valid even for supersonic flows ($M>1$) as long as $M\sin\lambda<1$. This case is of interest in modern compressors that operate at supersonic relative Mach numbers but whose axial Mach number is less than unity. However, in the latter situation, Γ is negative as $|n|\to\infty$ and Eq. (4a) is less suitable for numerical computation because of the large number of terms (or propagating modes) that contribute to the sum. It is possible to extract certain terms from Eq. (4a) to greatly enhance its convergence characteristics in the supersonic case. This will be discussed later.

Observe that Γ is a quadratic function of $(2\pi n-\sigma)$. In the purely subsonic case, this quadratic opens upward and assumes a negative value at the origin. On the other hand, at the point where the sgn function in Eq. (4a) changes sign, Γ is positive. It is obvious then that whenever Γ is negative, so is the sgn function, so that this factor in Eq. (4a) may be replaced by (-1). In this case the trigonometric functions may be combined to form $\exp[-i|\bar{y}|(-\Gamma)^{1/2}/s\beta]$; this exponential together with $\exp(i\omega t)$ forms an outgoing wave in $\bar{y}$ space. That this must be so is clear from the Sommerfeld radiation condition. The situation is quite different in the supersonic case ($M>1$, $M\sin\lambda<1$). After reasoning along the previous lines, it is found that when Γ is negative, sgn will assume both positive and negative values in Eq. (4a). Therefore, the trigonometric functions will combine to form both $\exp[\pm i|\bar{y}|(-\Gamma)^{1/2}/s\beta]$ and the Sommerfeld radiation condition cannot be satisfied in supersonic flows.

Now Eq. (4a) represents the long time ($t\to\infty$) pressure field of an array of pulsating sources that obey Eq. (1). The y derivative of Eq. (4a) gives the pressure field of an array of vertical doublets; it is possible to show after considerable algebra that this y derivative is equivalent to Eq. (34) of Kaji and Okazaki after an obvious change in notation is made. This equivalence is valid in the purely subsonic case. An easier way to establish the correspondence between the current results and those of Ref. 1 is to observe that, apart from some factors of proportionality, I_3 of Kaji and Okazaki gives the pressure field of an array of sources. In fact, the last member of Eq. (19) of Ref. 1 is equivalent to the present result (4a).

In the supersonic leading edge case, the series representation of the pressure given by Eq. (4a) is only conditionally convergent. Nevertheless, it is possible to enhance greatly the convergence characteristics of the above series. Observe that as $|n|\to\infty$,

$$(-\Gamma)^{1/2}\cong\beta|2\pi n|\left\{1-\left[\left(\sigma+\frac{\omega s}{c_\infty}\frac{M\cos\lambda}{\beta^2}\right)\Big/2\pi n\right]+\ldots\right\} \tag{5}$$

so that by adding and subtracting series of the type

$$\sum_{n=1}^{\infty}\frac{\substack{\sin n\mu\\ \cos n\nu}}{n^{\epsilon}};\quad \epsilon=1 \text{ and } 2$$

for suitable choices of μ and ν, Eq. (4a) can be made absolutely convergent with speed n^{-3}. Note that the values of μ and ν may be deduced from Eq. (5) and the asymptotic behavior of the summand in Eq. (4a) as $|n|\to\infty$. It is not known whether the present approach offers an advantage (over, for example, Ref. 5) since numerical calculations have not been carried out as yet.

†$\mathrm{sgn}(x)=\pm1$ according to x positive or negative.

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Hamilton's Law and the Stability of Nonconservative Continuous Systems

80006

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Introduction

THE law of varying action is applied to obtain direct analytical solutions to nonconservative follower force systems. The solution to the Beck problem[1] is illustrated explicitly because it happens to be well known and is one of the relatively few such systems to which an exact solution may be obtained for comparison of results.

Hamilton's law of varying action,

$$\delta\int_{t_0}^{t_1}(T+W)\,\mathrm{d}t-\Sigma\frac{\partial T}{\partial\dot{q}_i}\delta q_i\Big|_{t_0}^{t_1}=0 \tag{1}$$

has now been applied, contrary to statements contained in Refs. 2 and 3, to generate the spacetime path and/or configuration of nonconservative systems for which the work cannot be expressed as a potential function.[4] The application is characterized by the same simplicity and accuracy as demonstrated for other conservative and nonconservative systems.[5-8]

Problem Formulation

When it is specified that the reference coordinate system (shown in Fig. 1) moves with, at most, constant velocity and that there are no discrete particles or rigid bodies associated with the deformable body, the kinetic energy of the body relative to the reference frame is simply $T=\int_V(\rho/2)(\dot{u}^2+\dot{v}^2+\dot{w}^2)\mathrm{d}V$, where u, v, and w are the displacement components of a material point of the body.

Insight into the meaning of Hamilton's law and Hamilton's principle[5,8] permits the work function to be expressed simply as the sum of the products of the internal and external forces with their respective associated displacements, regardless of what the forces may be a function; in the physical system.[9] For a deformable body with a stress field σ, continuous body forces B, continuous body moments $\bar{M}$, continuous surface forces $\bar{S}$, continuous surface moment M, discrete body and/or surface forces, and discrete body and/or surface moments, the work function (not the work)[9] for application of

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